

A tree-growth model to optimize silviculture

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Abstract : In this paper, we present the description of a simplified model of the dynamic of a mono-specific even-aged forest. The model studied is a tree-growth model based on a system of two ordinary differential equations concerning the tree basal area and the number of trees. The analytical study of this model permits us to predict the behavior of the system solutions. We are trying to highlight the influence of economic parameters and growth parameters on the system solutions, in the framework of the optimization of silviculture.

Keywords : growth model ; optimization ; control

1 Introduction

The forest management, because of its impact on our environment, is a topic that now involves researchers of many disciplines : forestry, economy, and ecology. These various communities have models adapted to the questions they wish to tackle. The economists usually study the best age at which to cut down a tree or stand of trees, the simultaneous management of several forest stands. As to foresters, they are moreover interested in silviculture at the stand level. We will focus on the models developed by foresters.

The models of growth for silviculture, expanded rapidly these last few years, and represent a significant part of the developed models. Here, we are focusing on a particular type of forest (thus a particular type of model) : a mono-specific even-aged forest where all trees belong to the same species and are the same age.

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Models built by forest modellers are based on statistical adjustments of dendrometric data [1] : these models accurately describe the evolution of a forest, but the analytical study of those models is made difficult due to their complexity. The analytical study of models allows us to predict the influence of different parameters : for instance, if by modifying parameters to take into account the climate change and analyzing so the potential consequences is provide. To allow analytical studies while having realistic model, we decide in this paper to consider a simplified model.

The models, on which we are focusing here, are tree centered distance independent models where the trees are not spatialized. In this type of model each tree is characterized by its basal area at the height of 1.3 meters : s and eventually by its height h . The model here described is based on the concepts developed in the growth model “Fagacées” [2] [3] for Oak or Beech forest. In this model, the link between the stand level and the individual tree level is explicit. This modeling allows us to describe the evolution of a forest of high density. “Fagacées” was broadcast through the project Capsis [4].

In order to allow analytical studies, we are starting with the simplified hypothesis in which we consider that all trees have the same basal area. We then consider a forest of n trees with basal area s which needs management such as thinning e throughout time.

The tree growth (due to the observed densities) is not independent of its neighbor’s growth. There is a competition for the available resources : photosynthesis and access to the light on one hand, and mineral nutrients on the other hand. Thus, cutting a tree implies the increase of its neighbors’ growth, and cutting no tree limits individual growth. That shows how a forest is not just a juxtaposition of trees.

This phenomenon is considered through an assessment equation that allows us to distribute the energy resources of the forest stand level between the various trees. This equation coupled to an equation describing the evolution of the number of trees leads us to a dynamic system of the forest. The studied model takes into account the characteristics that the foresters consider as the most important and as required : the basal area at 1.3 meters, the height, and the number of trees.

In Section 2 we will present the designing of the model. In Section 3 we will present general results on the behavior of the system solutions, then we will look for strategies which permit to leave the viability domain in minimal or maximal time. Finally in Section 4 we will highlight the influence of economic parameters and growth parameters for silviculture within the resolution of an optimization problem.

2 Designing the model

The trees density and the RDI of forest stand.

Let's consider a forest with a given area. It is intuitively clear that the tree number which this area can bear is limited. The environmental conditions (type of soil, local climatic conditions) are also factors to be taken into account for the maximum tree capacity. Foresters have established a law called “self-thinning”, described hereafter, to evaluate this maximum capacity. Let's note s_* the average tree basal area (at the height of 1.3 meters) of the forest. Reineke [5] observed monospecific forests with various densities and various species. Out of these observations he claimed the maximum tree number $n_{max}(s_*)$ that a stand can bear is given by the following self thinning relation :

$$\log n_{max}(s_*) = C_0 - \frac{q}{2} \log s_*,$$

where $C_0 > 0$ and $1 < q < 2$ are characteristic constant values of the forest species and of its environment, and in particular the ground fertility.

As for a given s_* , beyond the number $n_{max}(s_*)$ the trees die, the forest stand (in terms of tree number) has to remain under this limit. To simplify we will here make the assumption that all the trees have the same basal area s . For a forest of where the effective tree number is n , taking into account the relation of self-thinning, the density r is the ratio of the tree number and the maximum tree number of basal area s that the forest can support, r is then defined by :

$$r(n, s) = \frac{n}{n_{max}(s)}$$

is written this way :

$$r(n, s) = \frac{ns^{\frac{q}{2}}}{e^{C_0}} = A ns^{\frac{q}{2}}$$

where $A := e^{-C_0}$. This ratio r is called *RDI* (Relative Density Index or Reineke Density Index). By definition this ratio is always less than 1.

Competition between trees : from forest stand level to individual tree level

We will now describe the temporal evolution of state variables s , n and now r , of the considered forest.

The growth of a tree depends on its neighbors. There is competition for the resources and the death of a tree, natural or due to cuttings, implies an increased growth for its neighbors. We make the assumption, that in the course of time, silviculture makes it possible to maintain the trees uniformly distributed on the area. The model is characterized by the existence of two levels in the modeling.

At the forest stand level the available energy for the considered forest, is considered globally, due to photosynthesis or due to nutrients in the soil. This supplied energy makes it possible to ensure at the same time the maintenance and the growth of the trees. The share reserved for maintenance increases with the tree height, therefore with time, which limits all the more so the available part for growth. The energy left for growth is therefore a decreasing time function and allows the increase of basal area of the forest stand. The increase of basal area of the forest stand at its peak of density $r(n(t), s(t)) = 1$ is given by the function $V(\cdot)$. We assume that $V(t)$ verify the following properties :

$(H_1) : V(\cdot)$ is a positive, decreasing, convex function of t .

For a lower density ($r(n(t), s(t)) < 1$), the effective increase of the basal area for the forest is reduced by a factor dependent on this same density : $g(r(n(t), s(t)))$ at any time. Thus the energy actually used at time t is given by :

$$g(r(n(t), s(t)))V(t).$$

The function $g(\cdot)$ is supposed to satisfy the following properties :

$(H_2) : g(\cdot)$ is an increasing, concave function of r such that $g(r) > r$ for $r \in (0, 1)$, $g(0) = 0$, $g(1) = 1$.

The concavity of g is related to crown development in relation to basal area.

On the individual tree level, tree growth is characterized by the evolution of tree basal area and therefore by the evolution of the function $s(t)$: the instantaneous increase is thus $\frac{ds(t)}{dt}$. As mentioned in the hypothesis all trees have the same basal area, the total sum increase of basal areas of all trees is $n(t)\frac{ds(t)}{dt}$. This total increase is obtained from the available energy resources. We thus obtain the equation which describes the link between the forest stand level and the individual tree level :

$$g(r(n(t), s(t)))V(t) = n(t)\frac{ds(t)}{dt} \text{ for } n(t) > 0, \text{ for all } t$$

For any t , this enables us to establish the first dynamic equation of our model :

$$\frac{ds(t)}{dt} = \frac{g(r(n(t), s(t)))}{n(t)} V(t)$$

In addition, the evolution of the tree number depends on several factors. To permit analytical study of the model, we've decided to simplify and we suppose the only cause of tree mortality is due to fallings that foresters could operate. We noted $e(t)$ the instantaneous rate of trees cutting at time t . Thus the evolution of the tree number is given by :

$$\frac{dn(t)}{dt} = -e(t).$$

We wish to preserve a minimum tree number in the forest stand, which implies $n(t) \geq \underline{n} > 0$, for any t . Technologically and to ensure a provisioning not too irregular, the thinning rate is limited : $0 \leq e(t) \leq \bar{e}$, for any t .

The forest is therefore described using the two state variables s, n and its evolution follows the following dynamic :

$$(\mathcal{S}_0) \quad \begin{cases} \frac{ds(t)}{dt} = \frac{g(r(n(t), s(t)))}{n(t)} V(t) \\ \frac{dn(t)}{dt} = -e(t) \end{cases}$$

with the constraints $0 \leq e(t) \leq \bar{e}$, $n(t) \geq \underline{n}$, $r(n(t), s(t)) = An(t)s(t)^{\frac{a}{2}} \leq 1$ for any t .

Foresters built this type of model from observed forest data. The available data only allows us to validate the model on a limited period of time. The system (\mathcal{S}_0) has therefore a time limit domain : $t \in [0, T_*]$.

It is a dynamic system in the state variables n and s , controlled by the control variable e . For a cutting policy, i.e. the data of a particular function $e(\cdot)$, and for each initial condition $(s(0), n(0))$, this system has a single solution : we will suppose the functions $g(\cdot), V(\cdot)$ are regular enough for it to happen. We will specify these trajectories in the following paragraph.

To finish with the model description, the tree height h is supposed to depend only on the tree basal area s and on the dominant height h_0 (average height of the 100 largest trees), h_0 is a concave function of time t and shouldn't depend on silviculture (cuttings in the course of time) and thus depends only on time t . The height h has therefore no influence on $s(t)$ and its evolution, h is consequently an output of the model.

If, as supposed earlier, the basal area s at time t is the same for all trees, the height h is also the same. We therefore deduce $h(t) = h_0(t)$, for any t .

3 Studiing the solutions

3.1 Model properties

The solutions of the dynamic system (\mathcal{S}_0) must satisfy in particular the constraint $r(n(t), s(t)) \leq 1$ for any t . If there is no cutting, i.e. if $e(t) = 0$ for any t , we deduce that $n(t) = n(0)$, $s(\cdot)$ and $r(\cdot)$ are increasing with respect to time t . Let's suppose there is one time $\tau < T_*$ such as $r(n(\tau), s(\tau)) = 1$, we deduced $t > \tau$ if we apply a control identically null then $r(n(t), s(t)) > 1$ and the constraint is no longer satisfied. In order to let us know which control we should apply we are led to study the evolution of the density function $r(\cdot)$:

$$\begin{aligned} \frac{dr(n(t), s(t))}{dt} &= r'_s(n(t), s(t)) \frac{g(r(n(t), s(t)))}{n(t)} V(t) - r'_n(n(t), s(t)) e(t) \\ &= \frac{r(n(t), s(t))}{n(t)} \left[\frac{q}{2} \frac{g(r(n(t), s(t)))}{s(t)} V(t) - e(t) \right] \end{aligned} \quad (1)$$

Out of this last equation we can deduce that in order to respect the constraint $r(n(t), s(t)) \leq 1$ for $t > \tau$, we should apply a non-identically null control on the system. Thus the cutting $e(t) = \frac{q}{2} \frac{V(t)}{s(t)}$ for $t > \tau$ respects the constraints by binding, i.e. $r(n(t), s(t)) = 1$. If we define the function $e_r(\cdot, \cdot)$ by : $e_r(s, t) := \frac{q}{2} \frac{V(t)}{s}$, for any $s > 0$, $t > 0$, the solutions, independently of the cutting function $e(\cdot)$ applied to the system, are only valid if the constraint : $e_r(s(t), t) \leq \bar{e}$ is satisfied. We are therefore led to formulate the following assumption (H_3) :

$$(H_3) : e_r(s_m(t), t) = \frac{q}{2} \frac{V(t)}{s_m(t)} < \bar{e} \text{ for all } t \in (0, T_*)$$

where $s_m(t)$ is the minimal value $s(t)$ can reach at the time t .

Remark 3.1 $s_m(t)$ is not specified at the moment but will be specified later on, however we can take an approximate lower bound for now : $s_m(t) > s(0)$.

We noted previously that the system (\mathcal{S}_0) is considered only for $t \in [0, T_*]$. It is advisable to specify now, the behavior of the solutions in this interval.

Definition 3.1 The function $\mathcal{V}(\cdot; \cdot)$ is defined by $\mathcal{V}(t; T) = \int_t^T V(u)du$ and represents the energy that has been available for growth in the period $[t, T]$.

The following Lemma shows us that the system validity field depends on this energy value :

Lemma 3.1 Assuming $(H_2), (H_3)$, then :

(i) if $\mathcal{V}(0; T_*)$ is large enough then there exists a time $\tau < T_*$ such that $r(n(\tau), s(\tau)) = 1$ and $n(\tau) = \underline{n}$. The dynamical system is only valid on the interval $[0, \tau]$. This time τ depends on the evolution of the cutting $e(\cdot)$.

(ii) conversely if $\mathcal{V}(0; T_*)$ is small enough then the dynamical system is valid throughout the entire interval $[0, T_*]$.

Proof : (i) From $g(r) \geq r$ we deduce : $\frac{ds(t)}{dt} \geq \frac{r(n(t), s(t))}{n(t)} V(t) = As(t)^{\frac{q}{2}} V(t)$ hence :

$$s(T_*)^{1-\frac{q}{2}} \geq s(0)^{1-\frac{q}{2}} + A(1 - \frac{q}{2})\mathcal{V}(0; T_*)$$

From $r(n(T_*), s(T_*)) \leq 1$ and $n(T_*) \geq \underline{n}$ we deduce : $s(T_*) \leq \frac{1}{(A\underline{n})^{\frac{2}{q}}}$. If $\mathcal{V}(0; T_*)$ is large enough, we obtain a contradiction.

(ii) Let's set τ the first period where $r(n(t), s(t))$ reaches 1, then for $0 < t \leq \tau$ we deduce : $\frac{ds(t)}{dt} \leq \frac{V(t)}{n(t)} \leq \frac{V(t)}{\underline{n}}$ and therefore $s(\tau) \leq s(0) + \frac{\mathcal{V}(0; \tau)}{\underline{n}}$. If $\mathcal{V}(0; T_*)$ is small enough, we deduce $r(n(\tau), s(\tau)) < 1$ in contradiction with the assumption. \square

Specific trajectories easily expressed in terms of control, will play an important role, we are introducing them here : let's consider the system of equations (\mathcal{S}_0) , for trajectory E_0 from a fixed initial condition $(s(0), n(0))$ we apply the maximum cutting $e(t) = \bar{e}$ until we reach the value n for the tree number, $t_{0,n}$ is the time needed to go from the tree number $n(0)$ to n . By definition, we therefore have $t_{0,n} = \frac{n(0) - n}{\bar{e}}$.

For trajectory E^0 , starting from the same initial condition (with $r(n(0), s(0)) < 1$) we apply the minimum cutting $e(t) = 0$ until reaching the value 1 for the RDI r , then we apply the control $e_r(s(t), t)$ until $n = \underline{n}$. t^0 is the time needed to go for the RDI from $r(n(0), s(0))$ to 1 and T^0 the final time. By definition, we therefore have t^0 and T^0 respectively solutions of :

$$\frac{q}{2}n(0)^{\frac{2}{q}-1}A^{\frac{2}{q}}\mathcal{V}(0;t^0) = \int_{r(n(0),s(0))}^1 \frac{u^{\frac{2}{q}-1}}{g(u)}du \text{ (at constant } n)$$

$$\underline{n}^{1-\frac{2}{q}} = n(0)^{1-\frac{2}{q}} + A^{\frac{2}{q}}(1 - \frac{q}{2})\mathcal{V}(t^0;T^0) \text{ (at constant } r)$$

Notations. To summarize we note the following definitions of the specific trajectories :

$$E_0 : e(t) = \begin{cases} \bar{e} & \text{if } t < t_{0,\underline{n}} \\ 0 & \text{if } t > t_{0,\underline{n}} \end{cases}$$

$$E^0 : e(t) = \begin{cases} 0 & \text{if } r(n(t), s(t)) < 1, \text{ i.e. } t < t^0 \\ e_r(s(t), t) & \text{if } t^0 < t < T^0 \end{cases}$$

For $t_{0,\underline{n}} < T < t^0$ we can also define an intermediate trajectory E_T

$$E_T : e(t) = \begin{cases} 0 & \text{if } t < T - t_{0,\underline{n}} \\ \bar{e} & \text{if } t > T - t_{0,\underline{n}} \end{cases}$$

$$\text{if } t_{0,\underline{n}} < T < t_{0,\underline{n}} + t^0$$

$$E_T : e(t) = \begin{cases} 0 & \text{if } t < t^0 \\ e_r(s(t), t) & \text{if } t^0 < t < t_* \\ \bar{e} & \text{if } t_* < t < T \end{cases}$$

$$\text{if } t_{0,\underline{n}} + t^0 < T < T^0$$

where t_* is defined by : $n(t_*)^{1-\frac{2}{q}} = n(0)^{1-\frac{2}{q}} + A(1 - \frac{2}{q})\mathcal{V}(t^0; t_*)$ and $(T - t_*)\bar{e} = n(t_*) - \underline{n}$.

We note that $E_{T^0} = E^0$ and by extension if $T < t_{0,\underline{n}}$ then $E_T = E_0$.

Figure 1 : Phase plane in the coordinates s and n .

The functions obtained by just following the trajectories E_0 , E_T and E^0 will be noted by the indices 0 , T and 0 .

From the increasing of the basal area s and the non increasing of the number n we deduce that the system solutions have no choice but to move to the right bottom in the phase plane. The Lemma 3.1 (i) has shown that, if $\mathcal{V}(0; T_*)$ is large enough, the solutions are not valid throughout the entire interval $[0, T_*]$. That implies that as from a time τ , the solution doesn't belong to the validity

domain defined by the constraints $r(n(t), s(t)) \leq 1$ and $n(t) \geq \underline{n}$. The only point which makes it possible to leave this validity domain is the point such as $r(n(\tau), s(\tau)) = 1$ and $n(\tau) = \underline{n}$, this point is represented by a square on Figure 1. As the solution remains valid basal area $s(t)$ verifies for all $t < T : s(t) \leq \bar{s} = \frac{1}{(A\underline{n})^{\frac{2}{q}}}$ (is deduced from $r(n(t), s(t)) \leq 1$).

We define \underline{T} (resp. \bar{T}) as the minimum (resp. maximum) time necessary to reach the point defined by $r(n(T), s(T)) = 1$ and $n(T) = \underline{n}$. Then :

- if $T \leq \underline{T}$ the solution remains valid whatever the trajectory (i.e. whatever the evolution of the cutting $e(\cdot)$).

- if $\underline{T} < T \leq \bar{T}$ the system has a solution on $[0, T]$ for certain controls $e(\cdot)$.

- if $T > \bar{T}$ the system has no solution on $[0, T]$ whatever the controls $e(\cdot)$.

We noted that, from Lemma 3.1, if $\mathcal{V}(0; T_*)$ is small enough then \underline{T} and especially \bar{T} can no exist.

A particular case

We could consider the particular function $g(r) = g_\theta(r) = r^{1-\theta}, 0 < \theta < 1$. In that case $\frac{ds(t)}{dt} = A^{1-\theta} \frac{s(t)^{\frac{q}{2}(1-\theta)}}{n(t)^\theta} V(t)$. The basal area s is explicitly deduced from the tree number n :

$$s(t)^{1-\frac{q}{2}(1-\theta)} = s(0)^{1-\frac{q}{2}(1-\theta)} + A^{1-\theta} (1 - \frac{q}{2}(1-\theta)) \int_0^t \frac{V(u)}{n(u)^\theta} du.$$

In this class of functions $g_\theta(\cdot)$ we consider the extreme case ($\theta = 0$) for which some of the properties of the hypothesis (H_2) are not satisfied : $g_0(r) = r$. In this last case $\mathcal{G}(r) \equiv 0$ and the evolution of the basal area s is independent from the evolution of the tree number n :

$$s(t)^{1-\frac{q}{2}} = s(0)^{1-\frac{q}{2}} + A(1 - \frac{q}{2})\mathcal{V}(0; t).$$

In this particular case, provided that $\mathcal{V}(0; T_*)$ is large enough, \underline{T} and \bar{T} are equal, don't depend on the cutting $e(\cdot)$ and are the unique solution of the following equation in T :

$$s(0)^{1-\frac{q}{2}} + A(1 - \frac{q}{2})\mathcal{V}(0; T) = \bar{s}^{1-\frac{q}{2}}.$$

3.2 Minimum and maximum time necessary to reach the point $(r, n) = (1, \underline{n})$

In order to succeed in the conclusion of the study of the minimum and maximum time needed to reach this point, we will need the following properties and definitions related to the function $g(\cdot)$. The increase in basal area of each tree is given by $\frac{g(r(n(t), s(t)))}{n(t)}V(t)$. We will know thereafter the evolution of $\frac{g(r(n(t), s(t)))}{n(t)}$, for the same aim, we will need to define the functions $\mathcal{G}(\cdot), \gamma(\cdot)$:

Definition 3.2 *The function $\mathcal{G}(\cdot)$ is defined by : $\mathcal{G}(r) = \frac{d}{dr}[\frac{r}{g(r)}] = \frac{g(r) - rg'(r)}{g^2(r)}$. The function $\gamma(\cdot)$ is defined by : $\gamma(r) = \frac{rg'(r)}{g(r)}$.*

In the “Fagacées” model, $g(r) = \frac{(1+p)r}{r+p}$ with $p > 0$, \mathcal{G} is constant $\mathcal{G}(r) \equiv G = \frac{1}{1+p}$.

From $\mathcal{G}(\cdot)$ and $\gamma(\cdot)$ definitions, we can establish the following properties for the model :

Lemma 3.2 *Assuming the hypothesis (H_2) , then :*

(i) *The function $\frac{r}{g(r)}$ is an increasing function of r and $\mathcal{G}(\cdot)$ satisfies $0 < \mathcal{G}(r)g(r) \leq 1$ for any $r > 0$.*

(ii) *The function $\frac{g(r(n, s))}{n}$ is a decreasing function of n .*

(iii) *The function $g(r(n, s))$ is an increasing function of s .*

(iv) *The function $\gamma(\cdot)$ satisfies $\gamma(r) \leq 1$ for any $r > 0$.*

(v) *if $n(t)$ and $s(t)$ are solutions of systems (\mathcal{S}_0) the function $\frac{g(r(n(t), s(t)))}{n(t)}$ is an increasing function of t .*

Proof : (i) From the concavity of $g(\cdot)$, $\frac{d}{dr}[g(r) - rg'(r)] = -rg''(r) > 0$ for any $r > 0$ and from $g(0) = 0$, we deduce that $g(r) - rg'(r) > 0$ and $\mathcal{G}(r) > 0$. From $g'(r) > 0$, $\mathcal{G}(r) \leq \frac{1}{g(r)}$.

$$(ii) \frac{\partial}{\partial n} \left[\frac{g(r(n, s))}{n} \right] = \frac{r(n, s)g'(r(n, s)) - g(r(n, s))}{n^2} = -\mathcal{G}(r(n, s)) \frac{g^2(r(n, s))}{n^2} < 0.$$

$$(iii) \frac{\partial g(r(n, s))}{\partial s} = g'(r(n, s))r'_s(n, s) > 0.$$

(iv) From $g(r) - rg'(r) > 0$ we deduce the result.

$$(v) \frac{d}{dt} \left[\frac{g(r(n(t), s(t)))}{n(t)} \right] = \frac{(g'(r)r'_s)(n(t), s(t))}{n(t)} \frac{ds(t)}{dt} + \mathcal{G}(r(n(t), s(t))) \frac{g^2(r(n(t), s(t)))}{n^2(t)} e(t)$$

and from (i) we deduce the result. \square

We are thus focusing on the trajectories and also on the strategies which allow us to reach respectively in a minimum and maximum time the point $(1, \underline{n})$ in the (r, n) coordinates.

The minimum time \underline{T} (resp. the maximum time \overline{T}) is reached by solving the problem of optimal control : $\min_{e(\cdot)} T$ (resp. $\max_{e(\cdot)} T$) with the set of admissible values for the control variable $[0, \bar{e}]$. $n(\cdot)$ and $s(\cdot)$ are the state variables governed by the system (\mathcal{S}_0) of initial condition $(n(0), s(0))$ and satisfying constraints, for all $t \in [0, T)$, $r(n(t), s(t)) \leq 1$, $n(t) \geq \underline{n}$ and the right end time constraints $r(n(T), s(T)) = 1, n(T) = \underline{n}$.

Proposition 3.1 Assume $(H_2), (H_3)$. If $n(\cdot)$ and $s(\cdot)$ are the solutions of the system (\mathcal{S}_t) for a control $e(\cdot)$ then, $\forall t \in [0, T]$:

$$(i) n_0(t) \leq n(t) \leq n^0(t)$$

$$(ii) s(t) \geq s^0(t)$$

$$(iii) \text{ if } g(r) = g_\theta(r) = r^{1-\theta}, s(t) \leq s_0(t)$$

If we assume that the final tree-number $n(T)$ is equal to \underline{n} then :

$$(iv) n(t) \leq n_T(t)$$

$$(v) \text{ if } g(r) = g_\theta(r) = r^{1-\theta}, s(t) \leq s_T(t)$$

Proof (i) and (iv) Follows from the definition.

$$(ii) \text{ From Lemma 3.2 (ii) we deduce : } \frac{ds(t)}{dt} = \frac{g(r(n(t), s(t)))}{n(t)} V(t) \geq \frac{g(r(n^0(t), s(t)))}{n^0(t)} V(t).$$

$$\text{For } t \leq t^0 \text{ we deduce : } \frac{ds}{g(r(n(0), s^{\frac{g}{2}}))} \geq \frac{V(t)}{n(0)} dt \text{ then by integration of the inequality :}$$

$$\int_{s(0)}^{s(t)} \frac{dx}{g(r(n(0), x^{\frac{q}{2}}))} \geq \frac{\mathcal{V}(0; t)}{n(0)} = \int_{s(0)}^{s^0(t)} \frac{dx}{g(r(n(0), x^{\frac{q}{2}}))} \text{ and we deduce } s(t) \geq s^0(t).$$

For $t > t^0$, $\frac{ds(t)}{dt} \geq \frac{V(t)}{n^0(t)} = \frac{ds^0(t)}{dt}$ and from $s(t^0) \geq s^0(t^0)$ we deduce by integration $s(t) \geq s^0(t)$.

(iii) From the previously stated expression of the basal area s and $n(t) \geq n_0(t)$ we deduce :

$$\frac{s(t)^{1-\frac{q}{2}(1-\theta)} - s(0)^{1-\frac{q}{2}(1-\theta)}}{A^{1-\theta}(1-\frac{q}{2}(1-\theta))} = \int_0^t \frac{V(u)}{n(u)^\theta} du \leq \int_0^t \frac{V(u)}{n_0(u)^\theta} du = \frac{s_0(t)^{1-\frac{q}{2}(1-\theta)} - s(0)^{1-\frac{q}{2}(1-\theta)}}{A^{1-\theta}(1-\frac{q}{2}(1-\theta))}$$

and hence the result.

(v) From the previously stated expression of the basal area s and $n(t) \leq n_T(t)$ we deduce :

$$\frac{s(t)^{1-\frac{q}{2}(1-\theta)} - s(0)^{1-\frac{q}{2}(1-\theta)}}{A^{1-\theta}(1-\frac{q}{2}(1-\theta))} = \int_0^t \frac{V(u)}{n(u)^\theta} du \geq \int_0^t \frac{V(u)}{n_T(u)^\theta} du = \frac{s_T(t)^{1-\frac{q}{2}(1-\theta)} - s(0)^{1-\frac{q}{2}(1-\theta)}}{A^{1-\theta}(1-\frac{q}{2}(1-\theta))}$$

□

If we remark that to reach the point $(r, n) = (1, \underline{n})$ in minimal time (resp. in maximal time) is equivalent to reach $s = \bar{s}$ in minimal time (resp. in maximal time), we deduce the trajectory that allows to reach the point $(r, n) = (1, \underline{n})$ in minimal or maximal time :

Corollary 3.1 Assume $(H_2), (H_3)$. Let \underline{T} the minimal time (resp. \overline{T} the maximal time) necessary to reach the point $(r, n) = (1, \underline{n})$ using the control $e(\cdot)$. Then :

(i) if \underline{T} is finite and $g(r) = r^{1-\theta}$, the trajectory that allows to reach the point $(r, n) = (1, \underline{n})$ in minimal time \underline{T} is the trajectory E_0 .

(ii) if \overline{T} is finite, the trajectory that allows to reach the point $(r, n) = (1, \underline{n})$ in maximal time \overline{T} is the trajectory E^0 . Maximal time \overline{T} is then the solution of :

$$\underline{n}^{1-\frac{2}{q}} - n(0)^{1-\frac{2}{q}} = A^{\frac{2}{q}}(1 - \frac{q}{2})\mathcal{V}(t^0; \overline{T}).$$

4 Optimization of silviculture

In order to optimize the silviculture, we are interested in problems which consist in seeking the minimal and maximum values of a variable function depending on the state variables n and s .

4.1 Preliminary results

We consider the hypothesis (H_4) :

(H_4) : there exists a constant $\underline{\gamma} > 0$ such as $\underline{\gamma} \leq \gamma(r)$ for any $r \in (0, 1)$.

We obtain the following Lemma (with proof in Annex A) :

Lemma 4.1 *Assume (H_2) , (H_3) . If $n(\cdot)$ and $s(\cdot)$ are the solutions of the system (S_t) for a control $e(\cdot)$ then, $\forall t \in [0, T]$: (with the convention that the inequalities including $s_0(t)$ are valid only if $g(r) = g_\theta(r) = r^{1-\theta}$)*

(i) *the function $\frac{g(r(n, s))}{n}$ satisfies :*

$$\frac{g(r(n^0(t), s^0(t)))}{n^0(t)} \leq \frac{g(r(n(t), s(t)))}{n(t)} \leq \frac{g(r(n_0(t), s_0(t)))}{n_0(t)}$$

Moreover, assume (H_4) :

(iia) *if $0 < b < \frac{1 - \frac{q}{2}\overline{\gamma}}{1 - \underline{\gamma}}$ then $n_0(t)s_0(t)^b \leq n(t)s(t)^b \leq n^0(t)s^0(t)^b$*

in particular, for $b = \frac{q}{2}$ the RDI $r(n, s)$ satisfies :

$$r(n_0(t), s_0(t)) \leq r(n(t), s(t)) \leq r(n^0(t), s^0(t))$$

(iib) *if $b > b_* = \frac{1 + \frac{q}{2}(\frac{1}{g(r(\underline{n}, s(0)))} - \underline{\gamma})}{1 - \overline{\gamma}}$ then $n^0(t)s^0(t)^b \leq n(t)s(t)^b \leq n_0(t)s_0(t)^b$*

(iii) *the relative increase ξ of the basal area s satisfies $\xi_m(t) = \frac{s^0(t)}{s^0(t)^{\frac{q}{2}} s^{1-\frac{q}{2}}} \leq \xi(t)$. Moreover,*

if $g(r) = g_\theta(r) = r^{1-\theta}$, $\xi_m(t) = \frac{s^0(t)}{s^0(t)^{\frac{q}{2}} s_0(t)^{1-\frac{q}{2}}} \leq \xi(t)$.

Remark : If we assume that the final tree-number $n(T)$ is equal to \underline{n} then the result obtained in the Lemma 4.1 remains valid if we replace respectively n^0, s^0 by n_T, s_T .

4.2 The optimization problem

We are focusing here, on the setting in the wood market of a forest whose evolution is set by the model studied in the previous paragraphs. We are introducing the price (minus the cost of thinning) which depends only on the basal area s and the height h : we noted $P_0(s, h, t)$. Owing to the fact that the height h does not depend on the basal area s and is equal to a fixed function h_0 of time t , the price can be written in a new function P of s and t : $P(s(t), t) := P_0(s(t), h(t), t)$.

In other words we will set the price function in the following form : $P_0(s, h, t) = p(s)he^{-\delta t}$ where $p(\cdot)$ is a price function for the basal area s and δ is the actualisation parameter. We deduce $P(s, t) = p(s)h_0(t)e^{-\delta t}$ and if we define the function $\delta_h(\cdot)$ by : $\delta_h(t) = \delta - \frac{h'_0(t)}{h_0(t)}$ for any $t > 0$ then $P'_t(s, t) = -\delta_h(t)P(s, t)$. Mostly to simplify we'll assume $p(s) = ks^\alpha, \alpha > 0$.

We are assuming that at each time t a quantity $e(t)$ is taken and that at the end of the period of exploitation T the remaining trees would have been cut. The instantaneous value of the trees that would have been cut is $P(s(t), t)e(t)$ and the final value is $P(s(T), T)n(T)$.

The criterion which we suggest to maximize consists of an integral term corresponding to the cuttings that would have occurred during the interval $[0, T]$ and the final term corresponding to the final cuttings at time T .

The optimization problem, relating to the cuttings $e(\cdot)$, on the interval $[0, T]$, is therefore written :

$$(\mathcal{P}) : \quad \max_{e(\cdot)} \int_0^T P(s(t), t)e(t)dt + P(s(T), T)n(T)$$

with $0 \leq e(t) \leq \bar{e}$ and n and s solutions of (\mathcal{S}_0) with initial conditions $(n(0), s(0))$ and fulfilling the constraints : $n(t) \geq \underline{n}$ et $r(n(t), s(t)) \leq 1$.

Intuitively, from the fact that the function $\frac{g(r(n, s))}{n}$ is a decreasing function of n (Lemma 3.2 (ii)), we are tempted to suggest the following assertion :

In order for the trees to get the best benefits from the nutrients, one should, from the beginning cut a significative number of trees, so that in the end of the exploitation timescale, one should get

a limited tree number of good quality.

We will try to validate or invalidate according to the cases this assertion and we will also try to answer the complementary yet important questions for management :

1) Does optimal silviculture depend on the term T ?

2) Which role the various parameters of the model play : economic parameters $p(\cdot)$, δ and growth parameters $g(\cdot)$, q ?

The optimization problem (\mathcal{P}) can be rewritten just by replacing $e(t)$ by $-\frac{dn(t)}{dt}$:

$$\max_{n(\cdot) \in \mathcal{C}} - \int_0^T P(s(t), t) \frac{dn(t)}{dt} dt + P(s(T), T)n(T)$$

where \mathcal{C} is the whole set of curves :

$$\mathcal{C} = \{n(\cdot) \in C^1([0, T]) \mid -\bar{e} \leq \frac{dn(t)}{dt} \leq 0 \text{ \& } An(t)s(t)^{\frac{q}{2}} \leq 1\}$$

By an integration by part we deduce :

$$\max_{n(\cdot) \in \mathcal{C}} \int_0^T \frac{dP(s(t), t)}{dt} n(t) dt + P(s(0), 0)n(0)$$

under the same constraints as in the initial problem.

We are here defining the function $\xi(\cdot)$, the relative increase of the basal area s , by $\xi(t) = \frac{s'(t)}{s(t)}$. By applying the results of Lemma 4.1 (iii) we deduce the following Proposition (with proof in Annex B) :

Proposition 4.1 Assume $(H_1), (H_2), (H_3), (H_4), T \leq \bar{T}$, then

(i) if $g(r) = r^{1-\theta}$, $\alpha > \alpha_* = 1 + (b_* - \frac{q}{2})(1 - \theta)$ and $\delta_h(t) \leq \alpha(1 - \theta)\xi_m(t)$, then the optimal trajectory is E_0 .

(ii) if $\alpha < \frac{1 - \frac{q}{2}\gamma}{1 - \gamma}$ and $\delta_h(t) \leq \alpha\underline{\gamma}\xi_m(t)$, then the optimal trajectory is E^0 .

From the remark following the Lemma 4.1 we deduce :

Corollary 4.1 *If we assume that the final tree-number $n(T)$ is equal to \underline{n} then :*

$$\text{if } \alpha < \frac{1 - \frac{q}{2}\gamma}{1 - \gamma} \text{ and } \delta_h(t) \leq \alpha \underline{\gamma} \xi_m(t), \text{ then the optimal trajectory is } E_T.$$

The condition on α in Proposition 4.1 (i) implies p must be sufficiently convex. Thus, under the conditions mentioned in the Proposition 4.1 (i) (if p is sufficiently convex and the parameter of actualization not too high), one may find it beneficial to cut the maximum tree number at the beginning to ensure a high rate for the remaining tree basal area at the end of T as foretold in the stated assertion. Similar results were obtained with the full model “Fagacées” [6]. Moreover, in the studied cases in Proposition 4.1, silviculture, i.e. cuttings policy $e(\cdot)$, does not depend on final time T . The conditions depend on economic parameters : a sufficient convexity of the price function relative to the basal area s and a small enough parameter of actualization. The stated assertion however is no longer satisfied under the conditions of the Proposition 4.1 (ii).

5 Conclusion

In that article, starting with a tree-growth model governed by the tree basal area and the number of trees, we study the viability properties of the system solutions. We highlighted the importance of the economic parameters and growth parameters on silviculture. Hence for a price (minus the thinning costs) is sufficiently convex and a parameter of actualization not too high, it is optimal to cut the trees at the beginning of the period of exploitation.

6 Proof of Lemma 4.1

(i) the result is a consequence of Lemma 3.2 (ii) (iii) and Proposition 3.1 (ii) (iii).

$$(ii) \frac{d(n(t)s^b(t))^a}{dt} = abn(t)^{a-1}s(t)^{ab-1}g(r(n(t), s(t)))V(t) - ae(t)n(t)^{a-1}s(t)^{ab}.$$

If we denote $y(n, s)$ the expression of $\frac{d(ns^b)^a}{dt}$, then, if we assume $0 < a < 1$:

$$\begin{aligned}
y'_n &= an(t)^{a-2}s(t)^{ab-1}(b[r g'(r) + (a-1)g(r)](n(t), s(t))V(t) + (1-a)e(t)s(t)) \\
&\geq abn(t)^{a-2}s(t)^{ab-1}(\underline{\gamma} + a - 1)g(r(n(t), s(t)))V(t) \\
y'_s &= abn(t)^{a-1}s(t)^{ab-2}([\frac{q}{2}r g'(r) - (1-ab)g(r)](n(t), s(t))V(t) - ae(t)s(t)) \\
&\leq abn(t)^{a-1}s(t)^{ab-2}(\frac{q}{2}\overline{\gamma} + ab - 1)g(r(n(t), s(t)))V(t)
\end{aligned}$$

hence, if we choose a such that $1 - \underline{\gamma} < a < \min(\frac{1 - \frac{q}{2}\overline{\gamma}}{b}, 1)$, we deduce $\frac{q}{2}\overline{\gamma} + ab - 1 < 0 < \underline{\gamma} + a - 1$ then $y'_n > 0$ and $y'_s < 0$ and :

$$\frac{d(n_0(t)s_0(t)^b)^a}{dt} = y(n_0(t), s_0(t)) \leq \frac{d(n(t)s(t)^b)^a}{dt} \leq y(n^0(t), s^0(t)) = \frac{d(n^0(t)(s^0(t))^b)^a}{dt}$$

by integration, we obtain the result.

(iib) From the expression of y'_n and y'_s and using $e(t) \leq e_r(s(t), t)$, we deduce that, if $a < 1 - \overline{\gamma}$:

$$\begin{aligned}
y'_n &\leq an(t)^{a-2}s(t)^{ab-1}(b(\overline{\gamma} + a - 1)g(r(n(t), s(t))) + \frac{q}{2}(1-a))V(t) \\
y'_s &\geq abn(t)^{a-1}s(t)^{ab-2}((\frac{q}{2}\underline{\gamma} + ab - 1)g(r(n(t), s(t))) - \frac{q}{2}a)V(t)
\end{aligned}$$

and, if $b > b_1(a) = \frac{q}{2} \frac{1-a}{1-a-\overline{\gamma}} \frac{1}{g(r(\underline{n}, s(0)))}$ then $y'_n < 0$.

if $b > b_2(a) = \frac{q}{2} \frac{1}{g(r(\underline{n}, s(0)))} + \frac{1 - \frac{q}{2}\underline{\gamma}}{a}$ then $y'_s > 0$.

To obtain the minimal limit value for b , as b_1 is increasing in a , and b_2 is decreasing in a , we choose the value a such that $b_1(a) = b_2(a)$, this value is $a_* = \frac{(1-\overline{\gamma})(1-\frac{q}{2}\underline{\gamma})}{1 + \frac{q}{2}(\frac{\overline{\gamma}}{g(r(\underline{n}, s(0)))} - \underline{\gamma})}$ hence the result if $b > b_* = b_i(a_*)$.

(iii) From Lemma 3.2 (i), $\frac{g(r)}{r}$ is a decreasing function of r then :

$\frac{s'(t)}{s(t)} = A \frac{g(r(n(t), s(t)))}{r(n(t), s(t))} \frac{V(t)}{s(t)^{1-\frac{q}{2}}} \geq A \frac{g(r(n^0(t), s^0(t)))}{r(n^0(t), s^0(t))} \frac{V(t)}{s_M(t)^{1-\frac{q}{2}}}$. From $s_M(t) \leq \bar{s}$ we deduce the result, if $g(r) = g_\theta(r) = r^{1-\theta}$, $s_M(t) = s_0(t)$. \square

7 Proof of Proposition 4.1

(i) Let's consider the auxiliary problem which consists in maximizing, at each time t , the integrand :

$$\begin{aligned}\frac{dP(s(t), t)}{dt}n(t) &= P'_s(s(t), t)g(r(n(t), s(t)))V(t) + P'_t(s(t), t)n(t) \\ &= kh_0(t)e^{-\delta t}(\alpha s(t)^{\alpha-1}g(r(n(t), s(t)))V(t) - \delta_h(t)n(t)s(t)^\alpha)\end{aligned}$$

We denote $y = ns^b$ and $z(y, s, t)$ by : $z(y, s, t) = \alpha s^{\alpha-1}g(Ays^{\frac{\alpha}{2}-b})V(t) - \delta_h(t)ys^{\alpha-b}$ then :

$$\begin{aligned}z'_y &= \alpha s(t)^{\alpha-1} \frac{[rg'(r)](n(t), s(t))}{y(t)} V(t) - \delta_h(t)s(t)^{\alpha-b} \\ &\geq s(t)^{\alpha-b-1}(\alpha(1-\theta) \frac{g(r(n(t), s(t)))}{n(t)} V(t) - \delta_h(t)s(t)) = s(t)^{\alpha-b}(\alpha(1-\theta)\xi(t) - \delta_h(t))\end{aligned}$$

From $\delta_h(t) < \alpha(1-\theta)\xi_m(t)$ we deduce $z'_y > 0$. Moreover if $b \geq \alpha$:

$$\begin{aligned}z'_s &= s(t)^{\alpha-2}(\alpha((\frac{q}{2}-b)rg'(r) + (\alpha-1)g(r))(n(t), s(t)))V(t) - (\alpha-b)\delta_h(t)y(t)s(t)) \\ &\geq \alpha s(t)^{\alpha-2}((\frac{q}{2}-b)(1-\theta) + \alpha-1)g(r(n(t), s(t)))V(t)\end{aligned}$$

Due to $\alpha > \alpha_*$ we can choose b such that $\max(b_*, \alpha) < b < \frac{\alpha-1}{1-\theta} + \frac{q}{2}$. Then $z'_s > 0$, from Lemma 4.1 (iib) we deduce the result.

(ii) We then denote $y = ns^\alpha$ and $z(y, s, t)$ by : $z(y, s, t) = \alpha s^{\alpha-1}g(Ays^{\frac{\alpha}{2}-\alpha})V(t) - \delta_h(t)y$.

$$\begin{aligned}z'_y &= \alpha s(t)^{\alpha-1} \frac{[rg'(r)](n(t), s(t))}{y(t)} V(t) - \delta_h(t) \\ &\geq \alpha \underline{\gamma} \frac{g(r(n(t), s(t)))}{n(t)s(t)} V(t) - \delta_h(t) = \alpha \underline{\gamma} \xi(t) - \delta_h(t)\end{aligned}$$

From $\delta_h(t) < \alpha \underline{\gamma} \xi_m(t)$, $z'_y > 0$. Moreover :

$$\begin{aligned}
z'_s &= \alpha s(t)^{\alpha-2} \left[\left(\frac{q}{2} - \alpha \right) r g'(r) + (\alpha - 1) g(r) \right] (n(t), s(t)) V(t) \\
&= \alpha s(t)^{\alpha-2} \left[\left(\left(\frac{q}{2} - \alpha \right) \gamma(r) + \alpha - 1 \right) g(r) \right] (n(t), s(t)) V(t)
\end{aligned}$$

From $\alpha < \frac{1 - \frac{q}{2}\gamma}{1 - \underline{\gamma}}$, we deduce : $z'_s < \alpha s(t)^{\alpha-2} \frac{1 - \frac{q}{2}}{1 - \underline{\gamma}} (\underline{\gamma} - \gamma(r)) g(r(n(t), s(t))) V(t) \leq 0$ then $z'_s < 0$. Moreover, from $z'_y > 0$, Lemma 4.1 (iia) with $b = \alpha$ and $z'_s < 0$ we deduce the result. \square

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